FLOW ABOUT IRREGULARITIES ON A FLAT SURFACE WITH THE FORMATION OF AN ATTACHED VORTEX

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The studies [1, 2] made a detailed investigation of the separation of flow over an uneven flat surface and the associated formation of a vortex attached to the surface. The Lavrent'ev scheme most closely approximates the actual motion of fluid about bodies. In accordance with this scheme, motion of the fluid is assumed to be uniformly vortical inside the vortex regions, while potential flow is assumed to exist outside these regions. Classical examples of flows conforming to this scheme – flow over a trench and a ledge [2] – have been examined only by numerical methods.

Here, we construct analytical solutions of this type of problem: flow over a projection and a trench in accordance with the Lavrent'ev scheme. It is assumed in both cases that the boundary between the vortex and the potential flow is an arc of a circle (a half-plane for the projection). The form of the projection and the channel depend on two parameters: $U(\Omega a)^{-1}$ and the angle α at which the arc approaches the surface (U is the velocity of the incoming flow, Ω is the vorticity constant, and a is half-chord which subtends the circle arc. Thus, the form may be arbitrary to a significant extent. The condition of existence of the vortex with a prescribed value of α is determined by the value of $U(\Omega a)^{-1}$. For the projection, the absolute value of this quantity must be less than a certain critical value approximately equal to 0.12. For the channel, the absolute value of $U(\Omega a)^{-1}$ must be greater than the critical value determined by α (as will be shown, this angle must not exceed 60°).

1. Flow Over a Projection. We will examine flow of the following form: a flow U which is uniform at infinity flows over a stationary vortex attached to a raised portion of a flat surface – a projection (Fig. 1). The flow inside the vortex is assumed to be uniformly vortical. Also, we will assume that the line separating the regions of vortex and potential motion (the upper boundary of the vortex) is a half-circle. We need to find the conditions for selecting the parameters of the vortex at which such a flow exists. We also need to find the form of the surface of the projection (the lower boundary of the vortex).

To obtain an analytical solution to the problem, we will use the method of joining steady uniformly vortical and potential flows that was proposed in [3]. Let the potential of the flow outside the vortex be known and be equal to $\Phi(\bar{W})$. The velocity field of the flow determined by this potential is written in the form

$$V = \frac{d\Phi(\overline{W})}{d\overline{W}}, \quad W = X + iY, \quad \overline{W} = X - iY; \tag{1.1}$$

while the expression for the streamlines has the form

$$\Phi(\overline{W}) - \overline{\Phi}(W) = -2i\psi, \ \overline{\Phi}(W) = \overline{\Phi(\overline{W})}. \tag{1.2}$$

Since the flow is steady, the boundary of the vortex coincides with one of the streamlines, such as ψ_0. The equation of the boundary is

$$\Phi(\overline{W}) - \overline{\Phi}(W) = -2i\psi_0. \tag{1.3}$$

A direct check shows that the expression for the velocity field of the vortex flow can be represented in the form

$$V = i\frac{\Omega}{2}(W - Z) + \frac{d\Phi(\overline{W})}{d\overline{W}}, \tag{1.4}$$

where Ω is the vorticity constant, while the function Z is found from the equation

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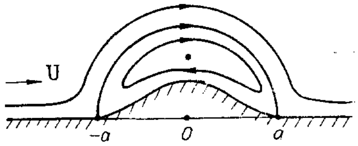

Fig. 1

$$\Phi(\overline{W}) - \overline{\Phi}(Z) = -2i\psi_0. \qquad (1.5)$$

It should be noted that the value of the function Z at the boundary of the vortex coincides with W, so that the velocity on the line which joins the flows is continuous.

With the above approach, it is evident that the character of the flow depends completely on the form of the potential $\Phi(\overline{W})$. In our case, when the line separating the vortex from the potential flow is a half-circle, the potential is conveniently chosen to be equal to the Zhukov function $\Phi(\overline{W}) = U(\overline{W} + a^2/\overline{W})$ (a is the radius of the half-circle). The streamlines are determined by the relation $U(\overline{W} + a^2/\overline{W}) - U(W + a^2/W) = -2i\psi$. The boundary of the vortex corresponds to $\psi_0 = 0$. Here, with allowance for (1.5), we can find $Z(\overline{W}) = a^2(\overline{W})^{-1}$. This means that the velocity in the vortex is

$$V = i\frac{\Omega}{2}W - i\frac{\Omega}{2}\frac{a^2}{\overline{W}} + U\left(1 - \frac{a^2}{\overline{W}^2}\right). \qquad (1.6)$$

The expression for the stream function which corresponds to this velocity field is written as

$$\psi = -\frac{1}{4}\left(|\overline{W}|^2 - a^2\right) + \frac{\Omega}{2}a^2\ln\frac{|\overline{W}|}{a} + UY\left(1 - \frac{a^2}{|\overline{W}|^2}\right),$$

where it was taken into account that $\psi = 0$ at $|\overline{W}| = a$. The equation of the lower boundary of the vortex is

$$-\frac{1}{4}\Omega\left(|\overline{W}|^2 - a^2\right) + \frac{\Omega}{2}a^2\ln\frac{|\overline{W}|}{a} + UY\left(1 - \frac{a^2}{|\overline{W}|^2}\right) = 0. \qquad (1.7)$$

Let us determine the conditions under which the above solution can exist. It follows from (1.6) that the velocity field of the vortex flow has a singularity at zero. This means that solution (1.6) makes physical sense only if the lower boundary of the vortex passes above the point $\overline{W} = 0$. The coordinate of the point at which the boundary of the vortex intersects the vertical axis is found from the solution of the transcendental equation

$$(Y^2 - a^2)\left(Y - \frac{4U}{\Omega}\right) = 2a^2Y\ln\frac{Y}{a}, \qquad (1.8)$$

obtained from (1.7) if we put X = 0. Equation (1.8) is convenient to study graphically. For a regime of flow about a projection with an attached vortex to exist, it is necessary that the graphs of the curves determined by the left and right sides of Eq. (1.8) have points of intersection in the interval (0, a). A sufficient criterion of the existence of such a solution is as follows:

$$-\frac{U}{\Omega a} \leqslant \frac{e^2+1}{4e(e^2-1)} \approx 0.12 \quad (e \simeq 2.71). \qquad (1.9)$$

It should be noted that the left side of inequality (1.9) is always positive — this follows from physical considerations (see Fig. 1). The quantity $U(\Omega a)^{-1}$ does not figure into the initial hydrodynamic equations, so that the result obtained regarding the dependence of the condition of existence of the attached vortex on this number is highly nontrivial. The physical significance of inequality (1.9) is fairly simple: the characteristic velocity in the vortex Ωa should be sufficiently large compared to the velocity of uniform flow.

The dimensionless number $U(\Omega a)^{-1}$ determines the angle at which the lower boundary of the vortex leaves the flat surface. For example, at the point $\overline{W} = a$ (see Fig. 1), $\theta = 4U(\Omega a)^{-1}$. It is evident from this that while the position of the upper boundary remains unchanged, the angle θ may change in relation to the flow parameters. The result obtained here is connected with the fact that the upper boundary of the vortex is normal to the surface. If it approaches the surface at the point $\overline{W} = a$ at an angle greater than a right angle, then,

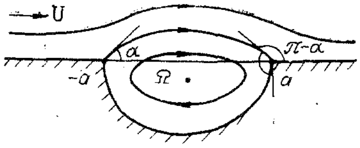

Fig. 2

as was shown in [4] (or see [2]), this angle should be equal to θ/2, i.e., the angle of departure of the lower boundary from the surface is unambiguously determined by the position of the upper boundary. This general property of the boundaries of attached vortices is illustrated below.

2. Flow over a Trench. We will examine the movement of a uniform flow over a stationary vortex which is attached to a cylindrical pit in a plane – a trench. We will assume that the upper boundary of the vortex is an arc of a circle and that the angle α at which this boundary leaves the surface (Fig. 2) is less than a right angle. We will use Eqs. (1.1)-(1.5) to analytically describe flow over the trench.

The potential of the flow outside the vortex is

$$\Phi(\overline{W}) = \frac{i2aU\pi}{\pi-\alpha}\left[1-\left(\frac{\overline{W}-a}{\overline{W}+a}\right)^{\pi/(\pi-\alpha)}\right]^{-1}. \tag{2.1}$$

This expression describes the potential motion of a uniform flow over a knoll in the form of a segment [1]. In the special case α = π/2, the potential (2.1) is equal to the Zhukov function to within an unessential constant term. The velocity field of the potential flow is found from the expression

$$V = 4a^2\beta^2U\frac{(\overline{W}^2-a^2)^{\beta-1}}{[(\overline{W}+a)^\beta-(\overline{W}-a)^\beta]^2}\quad\left(\beta=\frac{\pi}{\pi-\alpha}\right),$$

from which it is evident in particular that the velocity at infinity approaches the constant U.

In accordance wth (1.4) and (1.5), the velocity in the vortex is

$$V = i\frac{\Omega}{2}(W-Z) + 4a^2\beta^2U\cdot\frac{(\overline{W}^2-a^2)^{\beta-1}}{[(\overline{W}+a)^\beta-(\overline{W}-a)^\beta]^2}, \tag{2.2}$$

where the function Z is determined by the equation $[(Z-a)/(Z+a)]^\beta[(\overline{W}+a)/(\overline{W}-a)]^\beta = 1$. Of the set of solutions of this equation, we should choose a solution that ensures continuity of the velocity at the boundary where the flows are joined. It is not hard to see that this condition is satisfied by the following expression

$$Z = \frac{a^2}{\sin^2\frac{\pi}{\beta}}\left(\overline{W}+ai\,\mathrm{ctg}\,\frac{\pi}{\beta}\right)^{-1} + ai\,\mathrm{ctg}\,\frac{\pi}{\beta}. \tag{2.3}$$

It is then evident from (2.3) that Z = W for the points lying on the circle arc and that the function Z has a pole at the point $\overline{W} = -ai\cot(\pi/\beta)$. For the flow being examined to make physical sense, the lower boundary of the vortex must pass above this point.

We write the equation of the lower boundary in the form

$$\psi_* = 1-\frac{|\overline{W}|^2}{a^2}+\frac{1}{\sin^2\frac{\pi}{\beta}}\ln\left(\left|\frac{\overline{W}}{a}+i\,\mathrm{ctg}\frac{\pi}{\beta}\right|^2\sin^2\frac{\pi}{\beta}\right)+2\frac{Y}{a}\mathrm{ctg}\,\frac{\pi}{\beta}-\mathrm{Im}\,\frac{8\beta U}{a\Omega}\left[1-\left(\frac{\overline{W}-a}{\overline{W}+a}\right)^\beta\right]^{-1}=0. \tag{2.4}$$

The curve determined by this equation is symmetrical relative to the vertical axis. Near the point a, for example, $\psi_* \approx -\frac{8\beta U}{a\Omega}\left(\frac{q}{2a}\right)^\beta \sin\beta\delta,\ \beta<2$, where q and δ are, respectively, the modulus and the phase of the perturbation of the complex coordinate $\overline{W}$ at point a. It follows from this equality that the lower boundary leaves the surface at point a at the angle 2(π – α), i.e., the tangent to the upper boundary at point a is the bisector of this angle (see Fig. 2).

Let us find the condition of existence of the attached vortex in the trench. To do this, we need to find the coordinate of the point at which the lower boundary of the vortex intersects the vertical axis (the point of intersection must lie above the point $Y^0 = ia \cot(\pi/\beta)$. We introduce the new variables r and φ, given by the equality $(\bar{W} - a)/(\bar{W} + a) = re^{i\varphi}$. On the vertical axis, r = 1, and the point of intersection of the lower boundary with this axis $\overline{W}_* = ia \cot(\varphi_*/2)$ is determined by the equation

$$1 - \operatorname{ctg}^2 \frac{\varphi_*}{2} + \sin^{-2} \frac{\pi}{\beta} \ln\left(\left|\operatorname{ctg} \frac{\varphi_*}{2} + \operatorname{ctg} \frac{\pi}{\beta}\right|^2 \sin^2 \frac{\pi}{\beta}\right) - 2 \operatorname{ctg} \frac{\pi}{\beta} \operatorname{ctg} \frac{\varphi_*}{2} - \frac{4\beta U}{a\Omega} \operatorname{ctg} \frac{\beta\varphi_*}{2} = 0.$$

It is conveniently rewritten in the form

$$1 + \ln\left[\sin^2 \alpha \left(\operatorname{ctg} \frac{\varphi_*}{2} - \operatorname{ctg} \alpha\right)^2\right] - \left(\operatorname{ctg} \frac{\varphi_*}{2} - \operatorname{ctg} \alpha\right)^2 = \frac{4\beta U}{a\Omega} \sin^2 \alpha \operatorname{ctg} \frac{\beta\varphi_*}{2}. \tag{2.5}$$

Since we are examining the case of a vortex in a trench, the quantity Y_* must lie within the range from $-a \cot \alpha$ to zero. The values of φ_* corresponding to this interval satisfy the inequality

$$2\alpha < \varphi_* < \pi. \tag{2.6}$$

The conditions of existence of the attached vortex are readily explained by means of graphing. Inside the interval of φ_*, the left side of (2.5) is always negative. The curve of the dependence of the right side on φ_* is negative on the interval from 0 to $\pi - \alpha$ (remember that $U(a\Omega)^{-1}$ is negative). Having represented the path of these curves qualitatively, it is not hard to reach two important conclusions: the vortex exists in the trench only if the arc of the upper boundary of the vortex approaches the surface at the angle $\alpha < \pi/3$ (i.e., if the zero of the right side of (2.5) lies within the interval (2.6) or $\pi - \alpha > 2\alpha$); with the chosen α, $|U(a\Omega)^{-1}|$ should exceed a certain critical value $|U(a\Omega)_*^{-1}|$ determined by α. It should be noted that in contrast to the case of flow over a projection, an attached vortex is formed when the velocity of the uniform flow exceeds the critical value of the characteristic velocity in the vortex (the opposite result is valid for flow over a projection).

It follows from Eqs. (1.7) and (2.4) that the form of the projection and the trench can be arbitrary to a significant extent, depending on the values of $U(\Omega a)^{-1}$ and α (in the case of the trench). In the study of problems involving flow over irregularities of a specified form, this fact makes it possible to take the set of possible trench and projection profiles and choose the profile which is closest to that which is specified. It is evident from physical considerations (and numerical calculations) that a small amount of distortion of the profile may not significantly affect the character of the flow. Thus, the solutions obtained here can also be used for practically important problems of flow over irregularities of a prescribed shape involving the formation of an attached vortex.

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